The L-Framework

Structural Proof Theory in Rewriting Logic

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Logical Frameworks

Consider the following inference rule (tensor in Linear Logic):

$$\frac{\Gamma \vdash F \quad \Delta \vdash G}{\Gamma, \Delta \vdash F \otimes G} \otimes_R$$

Horn Clauses (Prolog)

```
prove Upsilon (F tensor G) :- split Upsilon Gamma Delta,
                              prove Gamma F, prove Delta G .
```

Rewriting Logic (Maude)

```
rl [tensorR] : Gamma, Delta |- F x G => (Gamma |- F) , (Delta |-G) .
```

Gap between what is represented and its representation

Rewriting Logic can rightfully be said to have "ϵ-representational distance" as a semantic and logical framework. (José Meseguer)

Where is the Magic ?

Rewriting logic: Equational theory + rewriting rules

- Propositional logic

```
op empty : -> Context [ctor] .
op _,_ : Context Context -> Context [assoc comm id: empty] .
eq F:Formula, F:Formula = F:Formula . --- idempotency
```

- Linear Logic (no weakening / contraction)

```
op _,_ : Context Context -> Context [assoc comm id: empty] .
```

- Lambek's logics without exchange

```
op _,_ : Context Context -> Context .
```

The general point is that, by choosing the right equations , we can capture any desired structural axiom. (José Meseguer)

Determinism vs Non-Determinism

Back to the tensor rule:

$$\frac{\Gamma \vdash F \quad \Delta \vdash G}{\Gamma, \Delta \vdash F \otimes G} \otimes_R \qquad \frac{\Gamma, F_1, F_2 \vdash G}{\Gamma, F_1 \otimes F_2 \vdash G} \otimes_L$$

Equations

Deterministic (invertible) rules that can be eagerly applied.

```
eq [tensorL] : Gamma, F1 * F2 |- G = Gamma, F1 , F2 |- G .
```

Rules

Non-deterministic (non-invertible) rules where backtracking is needed.

```
rl [tensorR] : Gamma, Delta |- F x G => (Gamma |- F) , (Delta |-G) .
```

Structural Properties of Sequent Systems

In good sequent systems the cut-elimination theorem holds, i.e., proofs using the cut rule can be transformed into cut-free proofs.

$$\frac{\Gamma \vdash \Delta, F \quad \Gamma, F \vdash \Delta}{\Gamma \vdash \Delta}\ cut$$

- Analytic proofs: (subformula property)
- Consistency.
- A cut-free system is more amenable for automatic reasoning.

How to prove cut-elimination for a given system?

It is often quite elaborated and exponentially exhaustive:

- Several proof obligations showing how cut permutes down.
- For that, it is useful to prove that some rules are invertible and some structural rules are admissible.

In this talk

Relying on rewrite and narrowing-based reasoning we introduce sufficient conditions and procedures for proving :

- admissibility of structural rules (weakening and contraction).
- invertibility of inference rules.
- permutability of inference rules (under certain conditions).
- cut-elimination of the system.

RL as a meta-logical framework in action.

The L-Framework (`https://carlosolarte.github.io/L-framework/`)

- A reflective implementation of our procedures in Maude.
- General enough for proving properties of different propositional systems: intuitionistic and classical logics, linear logic, and normal modal logics.

Outline

Rewrite Theories

A rewrite theory is the specification unit in rewriting logic

Definition (Rewrite Theory)

A tuple $\mathcal{R} = (\Sigma, E \cup B, R)$ consisting of:

- $(\Sigma, E \cup B)$ is an equational theory
- B is a set of structural axioms (assoc, comm, id).
- R is a set of labeled conditional rewrite rules $l \to r \texttt{ if } C$
- $(\Sigma, E \cup B)$ specifies states and deterministic computations
- R specifies dynamic, concurrent behaviors

RL proves sentences of the form $t \to t'$ ($\mathcal{R} \vdash t \to t'$) where $t, t' \in T_\Sigma(X)$.

$\to^1_{\mathcal{R}}$ is computable if the executability conditions hold ($\mathcal{R}$ is finite, equations in E are terminating and ground confluent, etc.)

Maude

- A high-performance rewriting logic engine
- A system module defines a rewrite theory $\mathcal{R}$.
- Executes admissible system modules (confluence and termination of *E*, coherence of *R* w.r.t. *E*, ...)
- Several generic formal analysis tools (rewrite, search, LTL model checker,etc).
- Reflective capabilities (RL as a meta-logical framework).

`http://maude.cs.illinois.edu/`

Plan

Sequent Systems

A sequent is an expression of the form $\Gamma \vdash \Delta$:

- Γ is the the antecedent and Δ the succedent.
- According to structural properties, Γ, Δ can be sets, multisets or lists of formulas.
- Δ can be a multiset (multi-conclusion) or restricted to one formula (single-conclusion)
- When Γ is empty it is called one-sided (otherwise, two-sided).

Inference Rules

$$\frac{S_1 \cdots S_n}{S}\ R$$

- S is the conclusion and $S_1 \cdots S_n$ the premises.
- If the premises are empty, then R is an axiom.

An example: Intuitionistic Propositional Logic

Syntax

$$F, G ::= p \mid \top \mid \bot \mid F \vee G \mid F \wedge G \mid F \supset G$$

Some of the rules of the system *G3ip* (single-conclusion, two-sided):

$$\frac{}{\Gamma, p \vdash p}\ I \qquad \frac{}{\Gamma \vdash \top}\ \top_R$$

$$\frac{\Gamma, F \vdash C \quad \Gamma, G \vdash C}{\Gamma, F \vee G \vdash C}\ \vee_L \qquad \frac{\Gamma \vdash F_i}{\Gamma \vdash F_1 \vee F_2}\ \vee_{R_i}$$

$$\frac{\Gamma, F \supset G \vdash F \quad \Gamma, G \vdash C}{\Gamma, F \supset G \vdash C}\ \supset_L \qquad \frac{\Gamma, F \vdash G}{\Gamma \vdash F \supset G}\ \supset_R$$

The OL in the L-Framework

We shall call Object Logic (OL) to the logical system we are analyzing.

Defining the syntax and inference rules of the OL is quite easy.

Generic constructs from the L-Framework

```
sort Formula .
sort MSFormula .                  --- Multiset of Formulas
op * : -> MSFormula .             --- Empty multiset
--- Multiset union
op _;_ : MSFormula MSFormula -> MSFormula [assoc comm id: * ] .
```

Particular instance for the OL

```
mod G3i is
...
 op _/\_ : Formula Formula -> Formula .
  --- Inference rules
  rl [AndL] :  F /\ G ; C |-- H => F ; G ; C |-- H .
  rl [AndR] :  C |-- F /\ G  =>  ( C |-- F) | ( C |-- G) .
  ...
endm
```

The OL in the L-Framework

A correct proof search procedure for free:

```
Maude> search [1] in G3i : p(1) /\ p(2) |-- p(2) /\ p(1) =>* proved .

Solution 1 (state 7)

Maude> show path 7 .
state 0, Sequent: p(1) /\ p(2) |-- p(2) /\ p(1)
===[ rl C ; F /\ G |-- H => F ; C ; G |-- H [label AndL] . ]===>
state 1, Sequent: p(1) ; p(2) |-- p(2) /\ p(1)
===[ rl C |-- F /\ G => (C |-- F) | (C |-- G) [label AndR] . ]===>
state 3, Goal: (p(1) ; p(2) |-- p(2)) | (p(1) ; p(2) |-- p(1))
===[ rl P ; C |-- P => proved [label I] . ]===>
state 5, Sequent: p(1) ; p(2) |-- p(1)
===[ rl P ; C |-- P => proved [label I] . ]===>
state 7, Goal: proved
```

Properties of inference rules

Definition

A rule

$$\frac{S_1 \cdots S_n}{S} \; R$$

is called:

1. admissible if S is derivable whenever $S_1, ..., S_n$ are derivable.
2. invertible if the rules $\frac{S}{S_1}, \cdots, \frac{S}{S_n}$ are admissible.

Proving Invertibility

The proof of invertibility proceeds by induction on the height of the derivation (and then, case analysis on the last rule applied).

Consider the proof o invertibility of $\vee_L$ and the case $\supset_R$:

$$\dfrac{\Gamma, F \vee G, A \vdash B}{\Gamma, F \vee G \vdash A \supset B} \supset_R$$

We can assume by induction (on a shorter derivation):

- $\Gamma, F, A \vdash B$
- $\Gamma, G, A \vdash B$

and then we conclude:

$$\dfrac{\Gamma, F, A \vdash B}{Goal1 : \ \Gamma, F \vdash A \supset B} \supset_R \quad \text{and} \quad \dfrac{\Gamma, G, A \vdash B}{Goal2 : \ \Gamma, G \vdash A \supset B} \supset_R$$

Admissibility of structural rules

We know that $\supset_R$ is invertible in *G3ip*. However, the case $\supset_L$ fails:

$$\frac{\Gamma, A \supset B \vdash A \quad \Gamma, B \vdash F \supset G}{\Gamma, A \supset B \vdash F \supset G} \supset_L$$

We know that:

- $\Gamma, A \supset B \vdash A$ (by hypothesis)
- $\Gamma, B, F \vdash G$ (by induction)

But this is not enough to complete the following figure:

$$\frac{\overset{??}{\Gamma, A \supset B, \mathrm{F} \vdash A} \quad \Gamma, B, F \vdash G}{Goal: \ \Gamma, A \supset B, F \vdash G} \supset_L$$

The admissibility of weakening is missing!

Admissibility of Structural Rules

The following structural rules are admissible in G3ip

$$\frac{\Gamma \vdash C}{\Gamma, F \vdash C}\ \mathrm{W} \qquad \frac{\Gamma, F, F \vdash C}{\Gamma, F \vdash C}\ \mathrm{C}$$

1. Admissibility of W is proved by induction on the height of the derivation.
2. The admissibility of C requires invertibility results. Consider, e.g., the case

$$\frac{\Gamma, F \vee G, F \vdash C \quad \Gamma, F \vee G, G \vdash C}{\Gamma, F \vee G, F \vee G \vdash C}\ \vee_L$$

By invertibility of $\vee_L$, we have Γ, F,F $\vdash C$ and, by induction, $\Gamma, F \vdash C$ as needed (the same for G).

Cut-Elimination

The following rule is admissible in G3ip

$$\dfrac{\begin{matrix}[\Pi] \\ \Gamma \vdash A\end{matrix} \quad \begin{matrix}[\Sigma] \\ \Gamma, A \vdash B\end{matrix}}{\Gamma \vdash B}\ Cut$$

Nested induction: on the complexity of A and subinduction on the sum of the heights of $[\Pi]$ and $[\Sigma]$.

Principal cases

$$\dfrac{\dfrac{\Gamma \vdash_n A \quad \Gamma \vdash_n B}{\Gamma \vdash_{s(n)} A \wedge B}\ \wedge_R \quad \dfrac{\Gamma, A, B \vdash_m C}{\Gamma, A \wedge B \vdash_{s(m)} C}\ \wedge_L}{\Gamma \vdash C}\ Cut \quad \rightsquigarrow \quad \dfrac{\Gamma \vdash A \quad \dfrac{\dfrac{\Gamma \vdash B}{\Gamma, A \vdash B}\ W \quad \overline{\Gamma, A, B \vdash C}}{\Gamma, A \vdash C}\ Cut}{\Gamma \vdash C}\ Cut$$

Cut-Elimination

The following rule is admissible in G3ip

$$\dfrac{\begin{array}{c}[\Pi] \\ \Gamma \vdash A\end{array} \quad \begin{array}{c}[\Sigma] \\ \Gamma, A \vdash B\end{array}}{\Gamma \vdash B}\ Cut$$

Nested induction: on the complexity of A and subinduction on the sum of the heights of $[\Pi]$ and $[\Sigma]$.

Non-principal cases

$$\dfrac{\dfrac{\Gamma, F, G \vdash_n A}{\Gamma, F \wedge G \vdash_{s(n)} A}\wedge_L \quad \Gamma, F \wedge G, A \vdash_{s(m)} B}{\Gamma, F \wedge G \vdash B}\ Cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, F, G \vdash_n A \quad \Gamma, F, G, A \vdash_{s(m)} B}{\Gamma, F, G \vdash B}\ Cut}{\Gamma, F \wedge G \vdash B}\wedge_L$$

Plan

Meta-theorems of sequent systems in RL:Invertibility

Consider invertibility of $\wedge_R$. We need to prove the following:

- Assuming that $\Gamma \vdash F \wedge G$ is derivable,
- Show that both $\Gamma \vdash F$ and $\Gamma \vdash G$ are derivable.

The proof needs to consider all the possible derivations for $\Gamma \vdash F \wedge G$:

Step 1

Unify the head of $\wedge_R$ with all the rules of the system.

$$\dfrac{?\Gamma_1 \vdash ?F_1 \quad ?\Gamma_1 \vdash ?G_1}{?\Gamma_1 \vdash ?F_1 \wedge ?G_1}\ \wedge_R \; + \; \dfrac{?\Gamma_1, ?F_2 \vdash ?C \quad ?\Gamma_1, ?G_2 \vdash ?C}{?\Gamma_2, ?F_2 \vee ?G_2 \vdash ?C}\ \vee_L$$

$$\rightsquigarrow$$

$$?\Gamma_3, ?F_2 \vee ?G_2 \vdash ?F_1 \wedge ?G_1$$

and consider the following ground sequent (freezing the variables):

$$\Gamma_3, F_2 \vee G_2 \vdash F_1 \wedge G_1$$

Step 2

Apply the rule $\vee_L$ on the ground sequent:

$$\frac{\Gamma_3, \boxed{F_2} \vdash F_1 \wedge G_1 \quad \Gamma_3, \boxed{G_2} \vdash F_1 \wedge G_1}{\Gamma_3, \boxed{F_2 \vee G_2} \vdash F_1 \wedge G_1} \vee_L$$

Step3

Collect all possible hypotheses by applying the theorem (just a rewriting rule!) to the sequents in the premises above.

Now we know, e.g. that the following sequents are derivable:

- $\Gamma_3, F_2 \vdash F_1 \wedge G_1$ (hypothesis above)
- $\Gamma_3, F_2 \vdash F_1$ (by induction)
- $\Gamma_3, F_2 \vdash G_1$ (by induction)

Step 4

Using all the hypotheses, find a proof of the goals.

The goal here is to prove the following ground sequents:

- $\Gamma_3, F_2 \vee G_2 \vdash F_1$ and
- $\Gamma_3, F_2 \vee G_2 \vdash G_1$

A search procedure with the System + Hypotheses is enough.

Meta-theorems of sequent systems in RL

Definition (Local Invertibility)

Let $\mathcal{S}$ be a sequent system and $\mathcal{H}$ be a (possibly empty) set of rules. Consider the following annotated inference rules:

$$\frac{k : S_1 \cdots k : S_m}{s(k) : S}\ r_s \qquad \frac{m : T_1 \cdots m : T_n}{s(m) : T}\ r_t$$

Under the assumption $\mathcal{H}$, the premise $l \in 1..m$ of the rule r_s *is height-preserving invertible relative to the rule r_t* iff for each $\theta \in CSU(s(k) : S, s(m) : T)$:

$$\mathcal{H} \cup \mathcal{R}_{\mathcal{S}} \cup \left\{\overline{(m : T_j)}\theta \mid j \in 1..n\right\} \cup \bigcup_{j \in 1..n} \{\overline{(m : S_l)}\gamma \mid \gamma \in CSU(k : S, \overline{(m : T_j)\theta})\} \Vdash \overline{(k : S_l)}\theta$$

Global view:

extended theory $\Vdash$ premise of s is provable

i.e., $S_i \rightarrow \texttt{proved}$ in the extended theory.

Meta-theorems of sequent systems in RL

Definition (Local Invertibility)

$$\frac{k : S_1 \cdots k : S_m}{s(k) : S}\, r_s \qquad \frac{m : T_1 \cdots m : T_n}{s(m) : T}\, r_t$$

Under the assumption $\mathcal{H}$, the premise $l \in 1..m$ of the rule r_s *is height-preserving invertible relative to the rule* r_t iff for each $\theta \in CSU(s(k) : S, s(m) : T)$

$$\mathcal{H} \cup \mathcal{R}_{\mathcal{S}} \cup \left\{\overline{(m : T_j)\theta} \mid j \in 1..n\right\} \cup \bigcup_{j \in 1..n} \{\overline{(m : S_l)\gamma} \mid \gamma \in CSU(k : S, \overline{(m : T_j)\theta})\} \Vdash \overline{(k : S_l)\theta}$$

We consider all possible sequents where both r_s and r_t can be applied.

Meta-theorems of sequent systems in RL

Definition (Local Invertibility)

$$\frac{k : S_1 \cdots \boxed{k : S_l} \cdots k : S_m}{s(k) : S}\ r_s \qquad \frac{m : T_1 \cdots m : T_n}{s(m) : T}\ r_t$$

Under the assumption $\mathcal{H}$, the premise $l \in 1..m$ of the rule r_s *is height-preserving invertible relative to the rule* r_t iff for each $\theta \in CSU(s(k) : S, s(m) : T)$:

$$\mathcal{H} \cup \mathcal{R}_{\mathcal{S}} \cup \left\{\overline{(m : T_j)\theta} \mid j \in 1..n\right\} \cup \bigcup_{j \in 1..n} \{\overline{(m : S_l)\gamma} \mid \gamma \in CSU(k : S, \overline{(m : T_j)\theta})\} \Vdash \boxed{\overline{(k : S_l)\theta}}$$

The goal is to prove all the premises of $\boxed{r_s}$.

Meta-theorems of sequent systems in RL

Definition (Local Invertibility)

Let $\mathcal{S}$ be a sequent system ...

$$\frac{k : S_1 \cdots k : S_l \cdots k : S_m}{s(k) : S}\ r_s \qquad \frac{m : T_1 \cdots m : T_n}{s(m) : T}\ r_t$$

Under the assumption $\mathcal{H}$, the premise $l \in 1..m$ of the rule r_s *is height-preserving invertible relative to the rule* r_t iff for each $\theta \in CSU(s(k) : S, s(m) : T)$:

$$\mathcal{H} \cup \mathcal{R}_{\mathcal{S}} \cup \left\{\overline{(m : T_j)\theta} \mid j \in 1..n\right\} \cup \bigcup_{j \in 1..n} \{\overline{(m : S_l)\gamma} \mid \gamma \in CSU(k : S, \overline{(m : T_j)\theta})\} \Vdash \overline{(k : S_l)\theta}$$

External theorems (rewrite rules) are assumed as well as the rules of the system.

Meta-theorems of sequent systems in RL

Definition (Local Invertibility)

$$\frac{k : S_1 \cdots k : S_l \cdots k : S_m}{s(k) : S}\; r_s \qquad \frac{m : T_1 \cdots m : T_n}{s(m) : T}\; r_t$$

Under the assumption $\mathcal{H}$, the premise $l \in 1..m$ of the rule r_s *is **height-preserving invertible relative** to the rule* r_t iff for each $\theta \in CSU(s(k) : S, s(m) : T)$:

$$\mathcal{H} \cup \mathcal{R}_{\mathcal{S}} \cup \left\{\overline{(m : T_j)}\theta \mid j \in 1..n\right\} \cup \bigcup_{j \in 1..n} \{\overline{(m : S_l)}\gamma \mid \gamma \in CSU(k : S, \overline{(m : T_j)}\theta)\} \Vdash \overline{(k : S_l)}\theta$$

The premises of r_t are assumed as provable (rules of the form $seq \rightarrow \texttt{proved}$)

Meta-theorems of sequent systems in RL

Definition (Local Invertibility)

$$\frac{k : S_1 \cdots \; k : S_l \; \cdots k : S_m}{s(k) : S}\, r_s \qquad \frac{m : T_1 \cdots \; m : T_j \; \cdots m : T_n}{s(m) : T}\, r_t$$

Under the assumption $\mathcal{H}$, the premise $l \in 1..m$ of the rule r_s *is* ***height-preserving invertible relative*** *to the rule* r_t iff for each $\theta \in CSU(s(k) : S, s(m) : T)$:

$$\ldots \bigcup_{j \in 1..n} \{ \overline{(m : S_l)\gamma} \mid \gamma \in CSU(k : S , \overline{(m : T_j)\theta})\} \Vdash \ldots$$

Inductive reasoning: If r_s can be applied on the premises of r_t, the resulting sequents S_l are provable (with the same height).

Meta-theorems of sequent systems in RL

Now we have to repeat the same procedure for all the possible rules in the system $\mathcal{S}$.

Theorem

Let $\mathcal{S}$ be a sequent system and r_s an inference rule in $\mathcal{S}$. If r_s is invertible relative to each r_t in $\mathcal{S}$, then r_s is height-preserving invertible in $\mathcal{S}$.

Admissibility of Structural Rules

Definition (Local admissibility)

Let $\mathcal{S}$ be a sequent system, $\mathcal{I}$ be a (possibly empty) set of rules, and $r_t \in \mathcal{S}$ and r_s be rules given by

$$\frac{k : T_1 \cdots k : T_n}{s(k) : T}\ r_t \qquad \frac{S_1}{S}\ r_s$$

The rule r_s *is height-preserving admissible relative to r_t in* $\mathcal{S}$ under the assumptions $\mathcal{I}$ iff assuming that $i : S_1$ is provable then, for each $\theta \in CSU(i : S_1, s(k) : T)$,

$$\mathcal{R}_{\mathcal{S}} \cup \{\overline{(k : T_j)\theta} \mid j \in 1..n\}_{\mathcal{I}} \cup \{(\forall \vec{x})(\overline{k\theta} : S_1 \rightarrow \overline{k\theta} : S)\} \Vdash \overline{(i : S)\theta}$$

Theorem

Let $\mathcal{S}$ be a sequent system and

$$\frac{S_1}{S}\ r_s$$

be an inference rule. If r_s is admissible relative to each r_t in $\mathcal{S}$, then r_s is admissible in $\mathcal{S}$.

Definition (Cut-elimination relative to two rules)

Let $\mathcal{S}$ be a sequent system and $\mathcal{H}$ and $\mathcal{I}$ a set of rules. Let

$$\frac{n : S_1 \cdots n : S_m}{s(n) : S}\ r_s \qquad \frac{k : T_1 \cdots k : T_n}{s(k) : T}\ r_t$$

be inference rules in $\mathcal{S}$. Under the assumptions $\mathcal{H}$ and $\mathcal{I}$, the cut rule is admissible relative to r_s and r_t iff for each $\theta \in CSU(S, \mathtt{lcut})$ and $\gamma \in CSU(T, \mathtt{rcut}\theta)$:

$$\begin{array}{c} \mathcal{H} \cup \mathcal{R}_{\mathcal{S}} \cup \textit{ind-F} \cup \textit{ind-H} \cup \\ \{\overline{(n : S_j)\gamma}, \overline{S_j\gamma} \mid j \in 1..m\}_{\mathcal{I}} \cup \{\overline{(n : T_j)\gamma}, \overline{T_j\gamma} \mid j \in 1..n\}_{\mathcal{I}} \end{array} \quad \Vdash \quad \overline{\mathtt{hcut}\gamma}$$

where the variables in S and T are assumed disjoint and

$$\begin{array}{lcl} \textit{ind-F} & = & \left\{(\mathtt{hcut} \to \mathtt{lcut} \mid \mathtt{rcut})[t/A] \;\mid\; t \prec \overline{A\gamma}\right\} \\ \textit{ind-H} & = & \{\mathtt{hcut} \to (\overline{n\gamma : \mathtt{lcut}} \mid \overline{s(n)\gamma : \mathtt{rcut}})[\overline{A\gamma}/A]\} \cup \\ & & \{\mathtt{hcut} \to (\overline{s(n)\gamma : \mathtt{lcut}} \mid \overline{n\gamma : \mathtt{rcut}})[\overline{A\gamma}/A]\} \end{array}$$

Cut elimination conditions

Definition (Cut-elimination relative to two rules)

Let $\mathcal{S}$ be a sequent system and $\mathcal{H}$ and $\mathcal{I}$ a set of rules. Let

$$\dfrac{n : S_1 \cdots n : S_m}{s(n) : S}\ r_s \qquad \dfrac{k : T_1 \cdots k : T_n}{s(k) : T}\ r_t$$

be inference rules in $\mathcal{S}$. Under the assumptions $\mathcal{H}$ and $\mathcal{I}$, the cut rule is admissible relative to r_s and r_t iff for each $\theta \in CSU(S, \mathtt{lcut})$ and $\gamma \in CSU(T, \mathtt{rcut}\theta)$:

$$\ldots\{\overline{(n : S_j)\gamma}, \overline{S_j\gamma} \mid j \in 1..m\}_{\mathcal{I}} \cup \{\overline{(n : T_j)\gamma}, \overline{T_j\gamma} \mid j \in 1..n\}_{\mathcal{I}} \quad \Vdash \quad \overline{\mathtt{hcut}\gamma}$$

$$\dfrac{\mathtt{lcut} \qquad \mathtt{rcut}}{\mathtt{hcut}}\ Cut$$

r_s is applied on the left premise and r_t is applied on the right premise.

Cut elimination conditions

Definition (Cut-elimination relative to two rules)

$$\frac{n : S_1 \cdots n : S_m}{s(n) : S}\ r_s \qquad \frac{k : T_1 \cdots k : T_n}{s(k) : T}\ r_t$$

$$\ldots \cup \textit{ind-F} \cup \textit{ind-H} \cup \ldots \quad \Vdash \quad \overline{\mathtt{hcut}\gamma}$$

$$\begin{aligned}
\textit{ind-F} &= \left\{(\mathtt{hcut} \to \mathtt{lcut} \mid \mathtt{rcut})[t/A] \mid t < \overline{A\gamma}\right\} \\
\textit{ind-H} &= \{\mathtt{hcut} \to (\overline{n\gamma} : \mathtt{lcut} \mid \overline{s(n)\gamma} : \mathtt{rcut})[\overline{A\gamma}/A]\} \cup \\
&\quad\ \{\mathtt{hcut} \to (\overline{s(n)\gamma} : \mathtt{lcut} \mid \overline{n\gamma} : \mathtt{rcut})[\overline{A\gamma}/A]\}
\end{aligned}$$

`inf-F` is instantiated with proper subterms.

`ind-H` is instantiated with shorter derivations.

Cut elimination conditions

Finally, we test all the possible combinations of rules.

Theorem

Let $\mathcal{S}$ be a sequent system and $\mathcal{H}$ and $\mathcal{I}$ be set of rules. If for each r_s and $r_t \in \mathcal{S}$ the cut-rule is admissible relative to r_s and r_t under the assumptions $\mathcal{H}$ and $\mathcal{I}$, then the cut-rule is admissible in $\mathcal{S}$.

Plan

Reflective Implementation

Reflection (`META-LEVEL` in Maude) allows to:

- manipulate the theory $\mathcal{R}_S$;
- check whether $\mathcal{R} \vdash s \to \texttt{proved}$
- Solve the unification problems, replace variables with fresh constants, etc.

The L-Framework provides the machinery needed to attempt proofs of:

- Admissibility of structural rules: W and C.
- Invertibility analyses.
- Permutability analyses (some restrictions are in order).
- Identity expansion.
- Cut-Elimination. Different cut-rules are already specified: multiplicative, additive, one-sided, dyadic, systems, etc.

Case studies

G3ip – Intuitionistic, two-sided, single-conclusion, propositional logic.

Invertibilities											Structural		$G3ip_W$	$G3ip_{+inv}$
I	$\vee_L$	$\vee_{R_i}$	$\wedge_L$	$\wedge_R$	$\top_R$	$\top_L$	$\bot_L$	$\supset_L$	$\supset_R$	$\supset_L^{pR}$	W	C	$\supset_R$	C
$\checkmark_T$	$\checkmark_T$	$\checkmark_F$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_F$	X	$\checkmark_T$	$\checkmark_T$	X	$\checkmark_T$	$\checkmark_T$

Multi-conclusion Propositional Intuitionistic Logic (mLJ)

Invertibilities										Structural		mLJ_{+inv}
I	$\vee_L$	$\vee_R$	$\wedge_L$	$\wedge_R$	$\top_R$	$\top_L$	$\bot_L$	$\supset_L$	$\supset_R$	W	C	C
$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_F$	$\checkmark_T$	X	$\checkmark_T$

Propositional classical logic

Invertibilities										Structural		$G3cp{+}inv$
I	$\vee_L$	$\vee_R$	$\wedge_L$	$\wedge_R$	$\top_R$	$\top_L$	$\bot_L$	$\supset_L$	$\supset_R$	W	C	C
$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	$\checkmark_T$	X	$\checkmark_T$

Linear logic: monadic and dyadic systems:

LL and D−LL								LL			D−LL		$D-LL_{+W_c}$
1	$\bot$	$\top$	$\otimes$	$\&$	$⅋$	$\oplus_i$	$!$	$?$	$?_C$	$?_W$	$?$	*copy*	$?$
$\surd_T$	$\surd_T$	$\surd_T$	$\surd_F$	$\surd_T$	$\surd_T$	$\surd_F$	$\surd_F$	$\surd_F$	$\surd_T$	$\surd_F$	X	$\surd_F$	$\surd_T$

Plus the following theorem: If $\vdash \Gamma, !F$ then $\vdash \Gamma, F$

Normal Modal Logics: K and S4

Invertibilities										Structural		Modal Rules			K_{+inv}	$S4_{+inv}$
I	$\vee_L$	$\vee_R$	$\wedge_L$	$\wedge_R$	$\top_R$	$\top_L$	$\bot_L$	$\supset_L$	$\supset_R$	W	C	k	4	T	C	C
$\surd_T$	$\surd_T$	$\surd_T$	$\surd_T$	$\surd_T$	$\surd_T$	$\surd_T$	$\surd_T$	$\surd_T$	$\surd_T$	$\surd_T$	X	$\surd_F$	$\surd_F$	$\surd_T$	$\surd_T$	$\surd_T$

Cut-Elimination

Cut-elimination theorems for:

- LJ (additive and multiplicative cut)
- mLJ (multiple conclusion system)
- LK (multiple conclusion)
- Modal systems K, T and S_4.
- MALL (one sided, multiplicative)
- LL with explicit $?W$ and $?C$. Two cut-rules that need to be simultaneously eliminated:

$$\frac{\vdash \Gamma, F \quad \vdash \Delta, F^{\perp}}{\vdash \Gamma, \Delta}\ Cut \qquad \frac{\vdash \Gamma, !F \quad \vdash \Delta, (?F^{\perp})^{n}}{\vdash \Gamma, \Delta}\ mCut$$

- LL dyadic system.

$$\frac{\vdash \Gamma : \Delta_1, F \quad \vdash \Gamma : \Delta_2, F^{\perp}}{\vdash \Gamma : \Delta_1, \Delta_2}\ Cut \quad \frac{\vdash \Gamma : !F \quad \vdash \Gamma, F^{\perp} : \Delta}{\vdash \Gamma : \Delta}\ Cut!$$

The L-Framework

`https://carlosolarte.github.io/L-framework/`

Plan

Concluding Remarks

- We gave sufficient conditions for proving structural properties of sequent systems and used RL as a meta-logical framework.
- Our approach is generic (mild restrictions are imposed on sequents) and modular (properties can be proved incrementally).
- Thanks to the reflective capabilities in Maude, the implementation was reasonable simple.
- RL provided a straightforward encoding for the inference system and properties of interest.
- Several (successful) test cases.

Future directions

- conditions for variants of sequent systems (e.g., nested and hyper-sequents).
- exporting Maude's proof objects to a proof assistant.
- proof-assistant like environment.

Thanks!