First Order Logic

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Lecture Plan

1. First Order Logic Syntax
2. First Order Logic Semantics
3. Models and Tautologies
4. Examples

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First Order Logic (FOL)

- Domain of discourse
- Variables range over specific domains such Integers or Reals
- Relations such as $\leq$ and functions like $\times$ and $+$
- Variables may be quantified universally or existentially
- Examples of formulae in FOL: $\forall x \in \mathcal{Z}.\ \exists y \in \mathcal{Z}.\ x + y = 0$
  $\forall x \in \mathcal{N}.\ x \neq x + 1$

Terms

- Terms are expressions formed using function symbols and variables

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Signatures

- Signature $\mathcal{G} = (V, F, R)$ with three disjoint sets:
  - Set of variable symbols V
  - Function symbols F
  - Relation symbols R
- Each function and relation symbol is associated an arity
  - For instance add has arity two
  - Constant symbols such as $zero$ or one are considered to be function symbols of arity 0

Formulae

- Formulas are formed from terms using relations

$$simpForm := rel(term, term, ..., term) \mid term \equiv term$$

$$\begin{aligned} form := \quad & simpForm \mid (form \wedge form) \mid (form \vee form) \mid \\ & (form \rightarrow form) \mid (\neg form) \mid \forall x.(form) \\ & \exists x.(form) \mid true \mid false \end{aligned}$$

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- For instance, $ge(one, zero) \vee ge(add(one, one), x))$ is a formula

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Semantics

- In order to interpret terms, we first need to assume a given domain $\mathcal{D}$ of values, for instance, integers or natural numbers
- A first order structure $\mathcal{S} = (\mathcal{G}, \mathcal{D}, \mathcal{F}, \mathcal{R}, f)$ is defined over a signature $\mathcal{G}$
  - $\mathcal{G} = (V, F, R)$
  - A domain $\mathcal{D}$ of discourse
  - A set of functions $\mathcal{F}$ (including constant symbols)
  - A set of relations $\mathcal{R}$
  - And a mapping $f : F \cup R \rightarrow \mathcal{F} \cup \mathcal{R}$

First Order Structure: Example

$$\mathcal{S} = (\,(\{x, y\}, \{add, mult\}, \{ge\}), \mathcal{N}, \{+, \times\}, \{\geq\}, f)$$

- f associates add to $+$ (addition over integers)
- f associates $mult$ to $\times$ (multiplication over integers)
- f associates ge to $\geq$

Semantic Interpretation: Terms

- An assignment $a : V \rightarrow \mathcal{D}$ is a mapping of variables to values in the domain $\mathcal{D}$
- A semantic interpretation $T_a : terms(\mathcal{G}) \rightarrow \mathcal{D}$ maps each term to a variable in the domain

$$T_a(v) = a(v), \text{ for } v \in V$$
$$T_a(func(e_1, e_2, ..., e_n)) = f(func)(T_a(e_1), T_a(e_2), ..., T_a(e_n))$$

Semantic Interpretation: Example

- $\mathcal{D}$ is the domain of Integers
- Assignment $a = \{x \mapsto 2,\ y \mapsto 3,\ z \mapsto 4\}$
- f maps add to the addition over Integers
- Therefore, $T_a(add(add(x, y), z)) = 5 + 4 = 9$

Semantic Interpretation: Formulae

Interpretation $M_a : forms(\mathcal{G}) \longrightarrow \{\text{TRUE}, \text{FALSE}\}$ assigns a truth value to each formula in $\mathcal{G}$

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$M_a(rel(e_1, ..., e_n)) = f(rel)(T_a(e_1), ..., T_a(e_n))$

$M_a(e_1 \equiv e_2) = (T_a(e_1) = T_a(e_2))$

$M_a(f_1 \wedge f_2) =$ TRUE iff $M_a(f_1) =$ TRUE and $M_a(f_2) =$ TRUE

$M_a(f_1 \vee f_2) =$ TRUE iff $M_a(f_1) =$ TRUE or $M_a(f_2) =$ TRUE

$M_a(f_1 \longrightarrow f_2) =$ TRUE iff $M_a(f_1) =$ FALSE or $M_a(f_2) =$ TRUE

$M_a(\neg f_1) =$ TRUE iff $M_a(f_1) =$ FALSE

$M_a(true) =$ TRUE

$M_a(false) =$ FALSE

Semantic Interpretation: Example

Interpreting the formula $\phi = ge(add(add(x, y), z), y) \wedge ge(z, zero)$ under the assignment $a = \{x \mapsto 2,\ y \mapsto 3,\ z \mapsto 4\}$ is done as follows:

- $T_a(add(add(x, y), z)) = 9$ as in the example before
- $T_a(y) = 3$
- $T_a(z) = 4$
- $T_a(zero) = 0$
- $M_a(ge(add(add(x, y), z), y)) = f(ge)(T_a(add(add(x, y), z)), T_a(y)) = 9 > 3 =$ TRUE
- $M_a(ge(z, zero)) = f(ge)(T_a(z), T_a(zero)) = 4 \geq 0 =$ TRUE
- $M_a(\phi) =$ TRUE

Semantic Interpretation

- Let a be an assignment, v a variable, and $d \in \mathcal{D}$
- A variant $a[d/v]$ is defined as

$$a[d/v](u) = \begin{cases} a(u) \text{ if } u \neq v \\ d \text{ if } u = v \end{cases}$$

$M_a(\forall v(\phi)) = \text{TRUE}$ iff for each $d \in \mathcal{D}$, $M_{a[d/v]}(\phi) = \text{TRUE}$
$M_a(\exists v(\phi)) = \text{TRUE}$ iff there exists $d \in \mathcal{D}$, $M_{a[d/v]}(\phi) = \text{TRUE}$

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Notations

- $a \models^{\mathcal{S}} \phi$ is read as a satisfies ϕ under the structure $\mathcal{S}$
- $\models^{\mathcal{S}} \phi$ is read $\mathcal{S}$ is a model of ϕ
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Formal Definition

- $a \models^{\mathcal{S}} \phi$ if $M_a(\phi) =$ TRUE
- $\models^{\mathcal{S}} \phi$ if $a \models^{\mathcal{S}} \phi$ for each assignment a
- $\models \phi$ if $\models^{\mathcal{S}} \phi$ for each structure $\mathcal{S}$

Examples

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2. $x \times 2 \equiv x + x$?
3. $(x \equiv y \wedge y \equiv z) \longrightarrow x \equiv z$?

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- $a \models^{\mathcal{S}} x \equiv y \times 2$, where $\mathcal{S}$ is the the structure that includes the domain of integers, and the functions $\times$ and $+$ are interpreted as the usual multiplication and addition over integers. This holds when a assigns x to 6 and y to 3

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- $\models (x \equiv y \wedge y \equiv z) \rightarrow x \equiv z$

Substitution

For a formula ϕ, term e, and variable v, $\phi[e/v]$ denotes

- Substitution of free occurrences of variable v in ϕ by e
- A free occurrence of v is not allowed to be replaced by a term that contains variables quantified in ϕ

Substitution

- $(\lambda x.\, \lambda y.\, y > x \;\rightarrow\; y > z)(y - 1)$ requires y to be renamed by some w such that $w \not\equiv y$, $w \not\equiv z$
- In $\forall y.(y > x \rightarrow y > z)$, variable x is not allowed to be replaced by the term $y - 1$

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- The graph is undirected, $\forall x. \forall y. (xRy \rightarrow yRx)$
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- The graph is undirected, $\forall x. \forall y. (xRy \rightarrow yRx)$
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- The graph contains cycles ?
- The graph is finite ?